ON BODIES OF MINIMUM WAVE DRAG

(O TELAKH MINIMAL'NOGO VOLNOVOGO SOPROTIVLENIIA)

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Certain variational problems are considered for a body that slightly disturbs a supersonic stream. It is found possible in the general case to separate the problem for the determination of the drag from the problem of the determination of the minimum drag body itself. For the solution of the first problem it is sufficient to express the properties of the body that are of interest (for example, forces, moments, volume, etc.) in terms of the values of the perturbation velocity potential on the characteristic surfaces enclosing the body. In this work, as an example, relations are found connecting the volume of the body with the values of the perturbation velocity potential on the characteristic surfaces enclosing the body.

It is shown that the perturbation velocity potential corresponding to flow past a body of minimum drag with arbitrary fixed leading and trailing sections and given volume satisfies, on the rear characteristic surface, Poisson's equation with mixed boundary conditions. Axisymmetric ducted bodies are found having minimum drag for fixed leading and trailing sections and given volume.

Also considered is the problem of the optimum choice of a fuselage having given length and volume with a given wing, and a lower estimate is obtained from the drag of the wing-fuselage system.

The method used in the work is that proposed by Nikol'skii for the solution of the problem of determining the contour of the body of revolution of minimum drag passing through two given points. We note that this method was used in the work [1] to find the drag of the optimum wing with a straight trailing edge perpendicular to the free stream, and in the work [2] for the solution of the problem of finding the drag of an optimum wing of arbitrary planform.

1. Formulation of the variational problem in supersonic flow. Let the following problem be given: to find the body possessing minimum wave drag at supersonic flight speeds and having certain quantities K_1, ..., K_n fixed (by quantities K we understand the overall dimensions of the body, its volume, the lift force or moment to which it is subjected, etc.).

We will solve this problem for linearized flow.

We find the envelope of all the characteristic surfaces that separate the perturbed and unperturbed parts of the flow in the direct and reversed streams (the direct stream has a velocity U_∞ in the undisturbed part, and the reversed has $-U_\infty$). Obviously these envelopes intersect along some line L_S. Let the volume enclosed between them be Ω_S and have area S.

We assume that a solution of the Goursat problem exists for the surface S (that is, the problem of determining within Ω_S a potential ϕ satisfying the wave equation with its values ϕ_S on S given). But then an arbitrary quantity K_i connected with the geometric or force properties of the body may be written in the form

$$K_i = B_i(\varphi_S) \qquad (1.1)$$

where B_i is a definite integro-differential operator. Consequently the variational problem posed above can be formulated as the problem of determining the potential ϕ_S corresponding to flow past a body of minimum drag under the conditions (1.1) (K_i = const). This circumstance is important in many cases, as it permits the problem of finding the extremal drag of a body to be separated from the problem of finding the body itself (which is associated with the solution of Goursat's problem), since many of the properties K_i (for example, forces, moments, volume) may be written down immediately on S. Thus an ordinary equation is found for determining ϕ_S and finding the drag of the extremal body. Example: the solution of the Goursat problem for the case of axial symmetry (Fig. 1) with $\phi = 0$ on AB is given by the formula

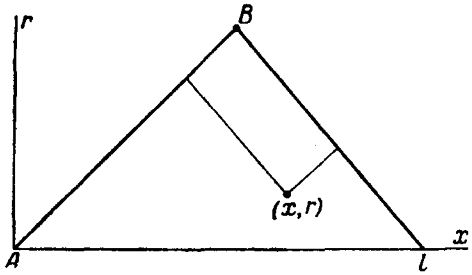

Fig. 1.

$$\varphi(x, r) = -\frac{2}{\pi\sqrt{l - x + \beta r}} \int_R^\Lambda \varphi'(\eta)\, K(\eta, x, r)\, d\eta \qquad \left(\Lambda = \frac{l}{2\beta},\ R = \frac{l - x + \beta r}{2\beta}\right)$$

$$K(\eta, x, r) = \frac{2\beta r \Pi(^1/_2\pi, n, k) - (l - x - \beta r) F(^1/_2\pi, k)}{\sqrt{x + \beta r - l + 2\beta\eta}} \qquad (\beta^2 = M^2 - 1) \qquad (1.2)$$

$$n = -\frac{x - \beta r - l + 2\beta\eta}{x + \beta r - l + 2\beta\eta}, \qquad k = \frac{(x - \beta r - l + 2\beta\eta)(l - x - \beta r)}{(x + \beta r - l + 2\beta\eta)(l - x + \beta r)}$$

Here M is the Mach number in the undisturbed stream, $\Pi(1/2\pi, n, k)$ is the complete elliptic integral of the third kind and $F(1/2\pi, k)$ is the complete elliptic integral of the first kind. Thus any variational problem may be solved in a stream possessing axial symmetry.

2. Basic relations. We will assume that the body has a cylindrical duct whose surface is parallel to $\mathbf{U}_\infty$; the body starts with a plane contour L_1 and ends with a plane contour L_2 (Fig. 2)*. We take an elementary contour on the surface S around the point (y, z) and draw through it the stream tube to its intersection with the plane $P(x = 0)$. We calculate the volume of this stream tube:

$$d\Omega = \int_L d\sigma\, dl = dq \int_L \frac{dl}{\rho V} \qquad (2.1)$$

Here dl is the length element of the stream tube, ds is its cross-sectional area, dq is the flow of gas across the stream tube, ρ is the density and V is the magnitude of the total velocity. Furthermore, regarding the perturbations produced by the body as small quantities of order ϵ, we have

$$\frac{1}{\rho V} = \frac{1}{\rho_\infty U_\infty}\left(1 + \frac{\beta^2}{U_\infty}\frac{d\varphi}{dl}\right) + O(\varepsilon^2), \qquad dl = dx + O(\varepsilon^2) \qquad (2.2)$$

$d\phi/dl$ is the derivative of the perturbation velocity potential along the stream line. Then

$$d\Omega = \frac{dq}{\rho_\infty U_\infty}\left[f(y, z) + \frac{\beta^2}{U_\infty}(\varphi_S - \varphi_P)\right] \qquad (2.3)$$

Here $x = f(y, z)$ is the equation of the surface S,

$$\varphi_S(y, z) = \varphi(x, y, z) \quad \text{at} \quad x = f(x, z) \qquad (2.4)$$

It is not difficult to see that

$$dq = \rho_\infty U_\infty dS\cos(nx) - \rho_\infty\left(\frac{\partial\varphi_S}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial\varphi_S}{\partial z}\frac{\partial f}{\partial z}\right)dS\cos(nx) \qquad (2.5)$$

where n is the outer normal to S. Integrating (2.3) and (2.4) over the entire characteristic surface S, we obtain

$$U_\infty\Omega_0 = \frac{1}{2}\iint_S\left(\frac{\partial\varphi_S}{\partial y}\frac{\partial f^2}{\partial y} + \frac{\partial\varphi_S}{\partial z}\frac{\partial f^2}{\partial z}\right)dS\cos(nx) - \beta^2\iint_S \varphi_S dS\cos(nx) \qquad (2.6)$$

$$\Omega_0 = \Omega_L + \Omega_{L1} - \Omega_{L2}$$

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* The assumption that the contours L_1 and L_2 are plane is not essential; the formula obtained in this section are valid also without this assumption.

Here Ω_L denotes the volume enclosed between the surface of the body and the surfaces x = const passing through the contours $L_1(P_{L1})$ and $L_2(P_{L2})$, Ω_{L1} is the volume between the cylindrical surface passing through the contour L_1 with its surface parallel to $\mathbf{U}_\infty$ and the planes P and P_{L1}, and Ω_{L2} is the volume enclosed between the cylindrical surface passing through the contour L_2 with its surface parallel to $\mathbf{U}_\infty$ and the planes P and P_{L2}.

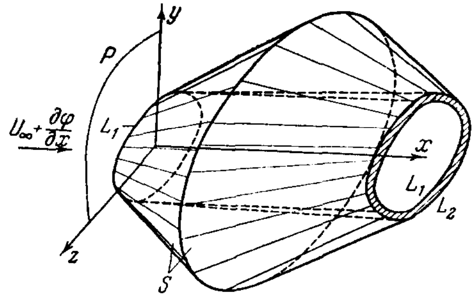

Fig. 2.

Applying the momentum law, we obtain for the force $\mathbf{R}$ acting on the body:

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = -\iint_S [\rho(\mathbf{V}\cdot\mathbf{n})\mathbf{V} + p\mathbf{n}]\,dS \qquad (2.7)$$

(p being the pressure) or, linearizing

$$\mathbf{R} = \mathbf{i}\frac{\rho_\infty}{2}\iint_S\left[\left(\frac{\partial\varphi_S}{\partial y}\right)^2 + \left(\frac{\partial\varphi_S}{\partial z}\right)^2\right]dS\cos(nx) -$$

$$- \mathbf{j}\rho_\infty U_\infty\iint_S\frac{\partial\varphi_S}{\partial y}\,dS(\cos nx) - \mathbf{k}\rho_\infty U_\infty\iint_S\frac{\partial\varphi_S}{\partial z}\,dS\cos(nx) \qquad (2.8)$$

Applying the equation of continuity to the surface S, we obtain

$$\iint_S dq = 0 \qquad (2.9)$$

or linearizing

$$-U_\infty\Sigma = \iint_S\left(\frac{\partial\varphi_S}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial\varphi_S}{\partial z}\frac{\partial f}{\partial z}\right)dS\cos(nx) \qquad (2.10)$$

where Σ is the difference in area between the boundary contours L_1 and L_2. If an undisturbed stream flows toward the body, then $\partial\phi/\partial y = \partial\phi_S/\partial z = 0$ on the forward characteristic surface and the integrations in formulas (2.6), (2.8) and (2.10) extend only over the rear characteristic surface.

3. Calculation of the drag of the extremal body. We formulate the variational problem. Let the forward and rear contours L_1 and L_2 be given, and also their volume, that is Ω_0. We introduce the forward and rear characteristic surfaces through the contours L_1 and L_2. We find the distribution of potential on the rear characteristic surface S for which the functional for the drag

$$X = \frac{\rho_\infty}{2} \iint_S (\varphi_{0y}^2 + \varphi_{0z}^2)\, dy\, dz$$

attains a minimum under the conditions*

$$\Omega_0 = \frac{1}{U_\infty} \iint_S f(\varphi_{0y} f_y + \varphi_{0z} f_z)\, dy\, dz - \frac{\beta^2}{U_\infty} \iint_S \varphi_0 dy\, dz = \text{const}$$

$$\Sigma = -\frac{1}{U_\infty} \iint_S (\varphi_{0y} f_y + \varphi_{0z} f_z)\, dy\, dz = \text{const}$$

$\phi = 0$ on the line of intersection of the forward and rear characteristic surfaces (indices y and z on ϕ_0 and f indicate differentiation with respect to y and z along the surface S). This problem is equivalent to the determination of the minimum of the functional (3.1)

$$I = \iint_S [\varphi_{0y}^2 + \varphi_{0z}^2 + 2\lambda_1 (\varphi_{0y} f_y + \varphi_{0z} f_z) + 4\lambda_2 f (\varphi_{0y} f_y + \varphi_{0z} f_z) - 4\lambda_2 \beta^2 \varphi_0]\, dy\, dz$$

where λ_1 and λ_2 are constant Lagrange multipliers. The minimum of the functional (3.1) is attained for ϕ_0 satisfying the following conditions (Fig. 3):

(1) $\Delta (\varphi_0 + \lambda_1 f + \lambda_2 f^2) = -2\lambda_2 \beta^2$ in region D

(2) $\varphi_0 = 0$ на l_1

(3) $\frac{\partial}{\partial n} (\varphi_0 + \lambda_1 f + \lambda_2 f^2) = 0$ на L_2

Here Δ is the Laplace operator, l_1 is the projection of the line of

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* As it is not difficult to see, the condition that the body passes through the given contours L_1 and L_2 is not included in these conditions. Such a condition could be formulated in the general cese only by knowing the solution of the Goursat problem. In certain cases (for example, plane or axisymmetric) it is realized automatically. With the realization of the conditions formulated in this section it is possible to guarantee that the body passes through either one of the contours L_1 or L_2.

intersection of the forward and rear characteristic surfaces on to a plane x = const, and D is the region between the contours L_2 and l_1.

Thus the perturbation velocity potential corresponding to flow past a body of minimum drag satisfies Poisson's equation with mixed boundary conditions on the rear characteristic surface.*

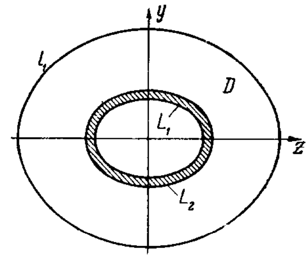

Fig. 3.

We calculate the values of the multipliers λ_1 and λ_2. For this we introduce functions ψ_1 and ψ_2 determined by the conditions

$$\triangle\psi_1 = 0, \quad \triangle\psi_2 = -2\beta^2 \text{ in } D$$
$$\psi_1 = f, \quad \psi_2 = f^2 \text{ on } l_1$$
$$\frac{\partial\psi_1}{\partial n} = \frac{\partial\psi_2}{\partial n} = 0 \quad \text{on } L_2$$

From the determination of the functions ψ_1 and ψ_2 it is evident that these functions depend only on β and the form of the surface S. Then the desired potential can be represented in the form

$$\varphi_0 = \lambda_1(\psi_1 - f) + \lambda_2(\psi_2 - f^2)$$

Formula (2.5), (2.8) and (2.10) for the body of minimum drag can, after simple transformations, be brought into the form

$$\Omega_0 = \frac{\lambda_1}{2U_\infty}\left[\int_{l_1} f^2 \frac{\partial\psi_1}{\partial n} dl - 2\beta^2 \iint_D \psi_1 dy\,dz\right] + \frac{\lambda_2}{2U_\infty}\left[\int_{l_1} f^2 \frac{\partial\psi_2}{\partial n} dl - 2\beta^2 \iint_D \psi_2 dy\,dz\right] \quad (3.2)$$

$$\Sigma = \frac{\lambda_1}{U_\infty}\left[\beta^2 S - \int_{l_1} f \frac{\partial\psi_1}{\partial n} dl\right] - \frac{\lambda_2}{U_\infty}\int_{l_1} f \frac{\partial\psi_2}{\partial n} dl \quad (3.3)$$

$$\frac{2X}{\rho_\infty U_\infty^2} = \frac{\lambda_1}{U_\infty}\Sigma - 2\frac{\lambda_2}{U_\infty}\Omega_0$$

We introduce the symbols

$$a = \beta^2 S - \int_{l_1} f \frac{\partial\psi_1}{\partial n} dl, \quad c = 2\beta^2 \iint_D \psi_2 dy\,dz - \int_{l_1} f^2 \frac{\partial\psi_2}{\partial n} dl$$

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* Here it is not demonstrated that a ϕ_0 satisfying the conditions enumerated above corresponds to the flow past any real body (that is, having everywhere positive thickness). In any case, for ϕ_0 determined in this way a lower estimate is obtained for the drag of a real body of minimum drag under the conditions formulated at the beginning of this section.

Strictly speaking, also in the works [1,2] lower estimates were obtained for the wave drag.

$$b = \int_{l_1} f \frac{\partial \psi_2}{\partial n} dl = \int_{l_1} f^2 \frac{\partial \psi_1}{\partial n} dl - 2\beta^2 \iint_D \psi_1 dy\, dz$$

Solving equations (3.2) and (3.3) with respect to λ_1 and λ_2, we obtain

$$\frac{\lambda_1}{U_\infty} = \frac{\Sigma}{a} + \frac{2b}{b^2 - ac}\Omega_1, \quad \frac{\lambda_2}{U_\infty} = \frac{2a}{b^2 - ac}\Omega_1 \left(\Omega_1 = \Omega_0 - \frac{b}{2a}\Sigma\right) \qquad (3.4)$$

For the drag X we will have

$$\frac{2X}{\rho_\infty U_\infty^2} = \frac{1}{a}\Sigma^2 + \frac{4a}{ac - b^2}\Omega_1^2 \qquad (3.5)$$

Together with the variational problem formulated at the beginning of this section, it is also possible to consider a more specialized problem. Let the leading and trailing contours L_1 and L_2 be given and the body of minimum drag be sought passing through these contours (that is, the volume of the body is arbitrary) [3]. For such a body the minimum drag is

$$\frac{2X}{\rho_\infty U_\infty^2} = \frac{1}{a}\Sigma^2, \qquad \Omega_1 = 0, \qquad \Omega_0 = \frac{b}{2a}\Sigma$$

In particular, if the line of intersection of the forward and rear characteristic surfaces lies in a plane x = const, then

$$\Omega_0 = -0.5l\,\Sigma \qquad (3.6)$$

where l is the length of the body. This formula permits calculation of the volume of the unknown body of minimum drag.

4. Body of revolution with cylindrical duct having minimum drag. As an example we consider the problem of determining a body of revolution with a cylindrical duct possessing minimum external drag. We will suppose that the following are given (Fig. 4): the volume of the body (that is, Ω_0), its length l, the radii R and R_1 of the leading and trailing sections, and the Mach number M. Special cases of this problem were considered in [1, 4-6].

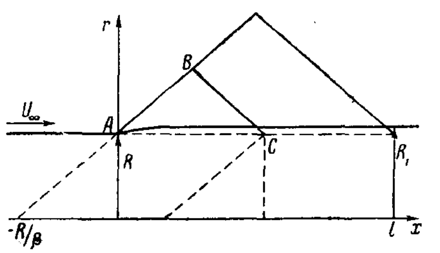

Fig. 4.

Henceforth we will neglect the quantity $\beta(R_1 - R)/l$ in comparison with unity. Solving the equations for ψ_1 and ψ_2 (cf. Section 3) and finding the values of the potential ϕ_0 corresponding to flow past a body of minimum drag, we obtain

$$\psi_1 = \frac{l}{2}, \quad \psi_2 = -\frac{\beta^2 r^2}{2} + \frac{\beta^2}{2}\left(\frac{l}{2\beta} + R\right)^2 + \beta^2 R^2 \ln \frac{2\beta r}{l + 2\beta R} + \frac{l^2}{4}$$

Henceforth, in this section we introduce the dimensionless quantities

$$\bar{\varphi}_0 = \frac{\varphi_0}{lU_\infty}, \qquad \bar{\lambda}_1 = \frac{\lambda_1}{U_\infty}, \qquad \bar{\lambda}_2 = \frac{\lambda_2 l}{U_\infty}, \qquad \bar{R} = \frac{\beta R}{l}$$

$$\bar{r} = \frac{\beta r}{l}, \qquad \bar{x} = \frac{x}{l}, \qquad \bar{\xi} = \frac{\xi}{l}, \qquad \bar{q} = \frac{q}{U_\infty l}, \bar{\Sigma} = \frac{\Sigma}{l^2}$$

and agree to drop the bars from the symbols. Then, for the potential ϕ_0, we obtain the following expression

$$\varphi_0 = \lambda_1\left[r - R - 0.5\right] + \lambda_2\left[R^2 \ln\frac{r}{R+0.5} - \frac{3}{2}r^2 + \right.$$
$$\left. + 2(1+R)r - \frac{1}{4}(R+0.5)(2R+5)\right]$$

From equations (3.2) and (3.3) it follows that

$$\lambda_1 = B\Sigma + A\Omega_1, \qquad \lambda_2 = -A\Omega_1$$

where

$$A = \frac{64}{\pi\{(1+4R)(1+4R-8R^2) + 64R^4 \ln[(R+0.5)/R]\}}$$

$$B = \frac{4}{\pi(1+4R)}, \qquad \Omega_1 = \frac{\Omega_0}{l^3} + 0.5\Sigma$$

Finally we have

$$\varphi_0 = B\Sigma\left[r - R - 0.5\right] - A\Omega_1\left[R^2\ln\frac{r}{R+0.5} - \right.$$
$$\left. -\frac{3}{2}r^2 + 2(R+0.5)r - \frac{1}{2}(R+0.5)^2\right] \tag{4.1}$$

Formula (3.5) permits the calculation of the drag of the extremal body which is for the time being still unknown:

$$\frac{2X}{\rho_\infty U_\infty^2 l^2} = B\Sigma^2 + 2A\Omega_1^2$$

We turn now to the solution of the Goursat problem. We find a distribution of sources on the interval $(-R,\ 1-R)$ of the x-axis such that potential takes the given values on the characteristic BC and is zero on AB. For this, advantage is taken of the solution of the wave equation known from linearized theory:

$$\varphi(r,\ x) = -\frac{1}{2\pi}\int_{-R}^{x-r} \frac{q(\xi)\,d\xi}{\sqrt{(x-\xi)^2 - r^2}} \tag{4.2}$$

For the intensity $q(\xi)$ of the sources we obtain the equation

$$-\frac{1}{2\pi}\int_{-R}^{R'} \frac{q(\xi)}{\sqrt{1+R-\xi}}\,\frac{d\xi}{\sqrt{1+R-2r-\xi}} = \varphi_0(r) \qquad (R' = 1+R-2r) \tag{4.3}$$

The solution of equation (4.3) is

$$q(\xi)=2\sqrt{1+R-\xi}\int_{\xi_1}^{R_1}\frac{\varphi_0'(r)\,dr}{\sqrt{2r-1+\xi-R}}\qquad \begin{pmatrix}R_1=R+0.5\\ \xi_1=0.5\,(1+R-\xi)\end{pmatrix}\tag{4.4}$$

On the basis of this formula the potential will be obtained inside the region bounded by the forward and rear characteristic cones that were introduced in Section 1. In the case of the body of minimum drag

$$q(\xi)=2B\Sigma\sqrt{(R+\xi)(1+R-\xi)}-\tag{4.5}$$

$$-A\Omega_1\left[(-1+2\xi)\sqrt{(R+\xi)(1+R-\xi)}+4R^2\operatorname{arc\,tg}\sqrt{\frac{R+\xi}{1+R-\xi}}\right]$$

In order to determine the shape of the body, formula (2.10) is used and applied to the forward contour ABC (Fig. 4). We then have

$$\Sigma(x)=2\pi\int_R^{r(x)}r\varphi_r\,dr\tag{4.6}$$

where $\Sigma(x)$ is the dimensionless area at section x and ϕ the value of the dimensionless potential on the characteristic BC.

Integrating formula (4.6) by parts and inserting the value of the potential (4.2) we obtain

$$\Sigma(x)=R\int_{-R}^{x-R}\frac{q(\xi)\,d\xi}{\sqrt{(x+R-\xi)-2R\,(x+R-\xi)}}+$$

$$+\int_R^{R+x}dr\int_{-R}^{x+R-2r}\frac{q(\xi)\,d\xi}{\sqrt{(x+R-\xi)(x+R-\xi-2r)}}$$

Changing the order of integration in the second integral and again integrating by parts, we then have

$$\Sigma(x)=\int_{-R}^{x-R}q'(\xi)\sqrt{(x-\xi)^2-R^2}\,d\xi\tag{4.7}$$

This last formula has a general character. In particular, if the radius of the duct is equal to $R=0$, then

$$\Sigma(x)=\int_0^x q'(\xi)(x-\xi)\,d\xi,\qquad \text{or}\qquad \frac{d}{dx}\Sigma(x)=q(x)$$

Substituting into formula (4.7) the value of $q'(\xi)$ from (4.5) and putting the integral so obtained into canonical form, we obtain finally

$$\Sigma(x) = \frac{B\Sigma}{2\sqrt{(x+2R)(1-x+2R)}} [2R(1+4R)\,\Pi(n, k) -$$
$$- 2R(x+2R)K(k) - (1-2x)(x+2R)(1-x+2R)E(k)] +$$
$$+ \frac{2}{3} A\,\Omega_1 \sqrt{(x+2R)(1-x+2R)}\,[(R+x-x^2-4R^2)E(k) -$$
$$- R(1-4R)K(k)] \tag{4.8}$$

where $K(k)$, $E(k)$ and $\Pi(n, k)$ are the complete elliptic integrals of the first, second and third kinds with parameters

$$k^2 = \frac{x(1-x)}{(x+2R)(1-x+2R)}, \qquad n = \frac{x}{x+2R}$$

Formula (4.8) is used also for the calculation of the shape of the body of minimum drag passing through the two given radii with arbitrary volume. In this case $\Omega_1 = 0$ [cf. (3.4) and (3.6)].

5. Investigation of a combination of bodies having minimum wave drag. Let the characteristic surface $S = S_1 + S_2$ consist of the inverse and direct Mach cones having vertices on the x-axis at the points $x = 0$ and $x = l$ (Fig. 5). Introducing in the plane x = const polar coordinates according to $y = r\cos\theta$, $z = r\sin\theta$ and noticing that

$$\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z} = \frac{\partial\varphi}{\partial r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\frac{\partial\psi}{\partial\theta}$$

it is possible to write in the following form the relations for the volume and for the area Σ of the body or system of bodies found inside the surface S:

$$\frac{U_\infty\Omega_0}{2\pi\beta^2} = -\iint\limits_{S_1}\frac{d\Phi}{dr}r^2dr + \iint\limits_{S_1}\Phi r dr + \iint\limits_{S_2}\frac{d\Phi}{dr}\left(r - \frac{l}{\beta}\right)r dr - \iint\limits_{S_2}\Phi r dr \tag{5.1}$$

$$\frac{U_\infty\Sigma}{2\pi\beta} = \iint\limits_{S_1}\frac{d\Phi}{dr}r dr - \iint\limits_{S_1}\frac{d\Phi}{dr}r dr \qquad \left(\Phi = \frac{1}{2\pi}\int_0^{2\pi}\varphi d\theta\right) \tag{5.2}$$

The quantity Φ we call the average potential. Thus, if the surface S consists of two Mach cones, then the volume of the body which is within this surface, and also the difference between the area of the entrance and exit sections, Σ, depends only on the value of the average potential Φ on the surface S.

$$\varphi = \Phi + \Delta \qquad \left(\int_0^{2\pi}\Delta d\theta = 0\right)$$

Then the formula for the drag can be rewritten as

$$X = -\pi\rho_\infty \int\limits_{S_1} \left(\frac{d\Phi}{dr}\right)^2 rdr + \pi\rho_\infty \int\limits_{S_2} \left(\frac{d\Phi}{dr}\right)^2 rdr - \frac{\rho_\infty}{2} \iint\limits_{S_1} \left[\left(\frac{d\Delta}{dr}\right)^2 + \right.$$

$$\left. + \frac{1}{r^2}\left(\frac{\partial\Delta}{\partial\theta}\right)^2\right] rdrd\theta + \frac{\rho_\infty}{2} \iint\limits_{S_2} \left[\left(\frac{\partial\Delta}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Delta}{\partial\theta}\right)^2\right] rdrd\theta \qquad (5.3)$$

Consequently, the expression for the drag X consists of two parts, of which one depends only on the quantity Φ and the other does not depend on this quantity.

We pose the following problem: let there be given some fixed bodies in a supersonic stream of gas, and let it be required to assemble a body of

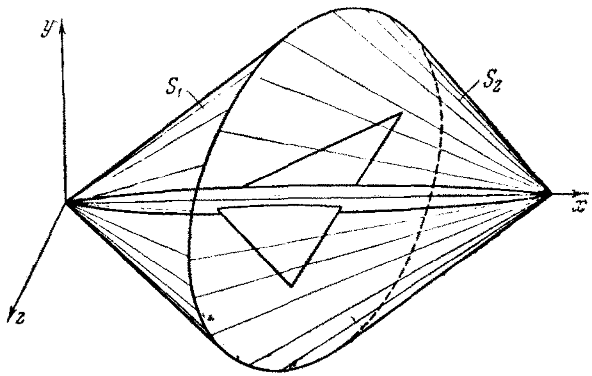

Fig. 5.

given length, area Σ and volume Ω_0, such that the drag experienced by the desired body and those bodies or their parts which are inside the surface S is a minimum (the duct, if the desired body contains one, is assumed to be circular). Relations (5.1), (5.2) and (5.3) are used for the solution.

It follows that all integrals over S_1 can be written down as given. The problem formulated above is a problem for the determination of the minimum of the functional (5.3) under the condition (5.1) and (5.2). Since conditions (5.1) and (5.2) depend only on the average potential, and the variable part of the expression for the functional X is represented in the form of two positive terms, of which one depends on Φ and the other does not depend on it, it is permissible to seek separately the minima of the variable parts:

$$I_1 = \iint\limits_{S_2} \left(\frac{d\Phi}{dr}\right)^2 rdr, \qquad I_2 = \iint\limits_{S_2} \left[\left(\frac{\partial\Delta}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Delta}{\partial\theta}\right)^2\right] rdrd\theta \qquad (5.4)$$

We find the minimum of I_1 under conditions (5.1) and (5.2). We note that by integration by parts, condition (5.1) may be transformed into the form:

$$\frac{U_\infty \Omega_0}{2\pi\beta^2} = -\iint_{S_1} \frac{d\Phi}{dr}\left(\frac{3}{2}r^2 - \frac{1}{2}r_0^2\right)dr + \iint_{S_2} \frac{d\Phi}{dr}\left(\frac{3}{2}r^2 - \frac{1}{2}r_0^2 - \frac{lr}{\beta}\right)dr \qquad (5.5)$$

Here r_0 is the radius of the duct. The function of Lagrange for the case considered is

$$L = \iint_{S_2} \left(\frac{d\Phi}{dr}\right)^2 r dr - 2\lambda_1 \iint_{S_2} \frac{d\Phi}{dr} r dr - 2\lambda_2 \iint_{S_2} \frac{d\Phi}{dr}\left(\frac{3}{2}r^2 - \frac{1}{2}r_0^2 - \frac{lr}{\beta}\right)dr \qquad (5.6)$$

where λ_1 and λ_2 are as yet undetermined constants.

The Euler equation for the functional (5.6) has the form

$$\frac{d\Phi}{dr} = \lambda_1 + \lambda_2\left(\frac{3}{2}r - \frac{1}{2}\frac{r_0^2}{r} - \frac{l}{\beta}\right) \qquad (5.7)$$

From the fact that the second variation of the quantity J_1

$$\delta^2 J_1 = 2\iint_{S_2} \left(\frac{d(\delta\varphi)}{dr}\right)^2 r dr$$

is always positive, we conclude that the expression (5.7) gives a minimum of the functional J_1. The constants λ_1 and λ_2 are found from conditions (5.1) and (5.2) analogously to Section 3.

It is easy to see that, if the problem were solved of determining the body of revolution of minimum drag with volume equal to the sum of the volumes of all the bodies (of the given ones and so also of the desired ones) inside the characteristic surface considered, with area Σ as for our body and having a potential on the forward part of the characteristic surface equal to the average of the desired potential, then the potential for the desired body of revolution on the rear part of the characteristic surface would agree with expression (5.7).

Such a body of revolution we will call an equivalent body of revolution. The problem of determining a body of revolution possessing minimum drag was studied in Section 4. We turn now to the second part of the problem.

That is, we find the minimum of the integral J_2. We assume that the fixed body is such that its potential is a function having derivatives and squares of derivatives that are integrable on S. This requirement is naturally always realized in practice.

We take first the case when in the plane $y = 0$ there is given a wing with symmetric profile and combined with it a fuselage symmetrical with respect to that plane.

Each function $\phi_k^{(i)}$ corresponds to a function $\Lambda_k^{(i)}$ where

$$\Delta_k^{(i)} = \varphi_k^{(i)} - \Phi_k^{(i)}$$

The potential from the wing on S_2 may be approximated by the function Φ_k' resolved into a Fourier series, so that the integrals

$$\iint_{S_2} \left\{\left[\frac{\partial(\Delta_k - \Delta_k')}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial(\Delta_k - \Delta_k')}{\partial \theta}\right]^2\right\} rdrd\theta$$

$$\iint_{S_2} \left[\frac{\partial(\Delta_k - \Delta_k')}{\partial r}\,\frac{\partial(\Delta + \Delta_k')}{\partial r} + \frac{1}{r^2}\,\frac{\partial(\Delta_k - \Delta_k')}{\partial \theta}\,\frac{\partial(\Delta + \Delta_k')}{\partial \theta}\right] rdrd\theta$$

(Δ corresponds to the potential, ϕ_0, of the fuselage) are as small as desired.

The potential of the fuselage satisfies the equation

$$-\beta^2 \frac{\partial^2\varphi_0}{\partial x^2} + \frac{\partial^2\varphi_0}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\theta^2} = 0 \qquad (5.8)$$

We resolve the quantity Φ_k' on S into a Fourier series. Taking the first n terms, we seek ϕ_0 in the form of a trigonometric polynomial in θ of degree n, so that on S_2

$$\varphi_{ki}' = -\varphi_{0i} \qquad (i = 1, \ldots, n)$$

that is, so that the sum of the corresponding Fourier coefficients for ϕ_k' and ϕ vanishes to order n.

The value of n may be chosen so that the integral

$$\iint_{S_2} \left[\left(\frac{\partial R_{kn}'}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial R_{kn}'}{\partial\theta}\right)^2\right] rdrd\theta$$

is sufficiently small (R_{kn}' is the remainder term of the Fourier series for the functions ϕ_k').

Thus it is found that, with the aid of a proper selection of the fuselage, generally speaking it is possible with a symmetrical wing-fuselage combination to obtain a drag differing as little as desired from the drag of the equivalent body of revolution by the deduction of the integral of the quantity Δ over S_1. At the same time this drag, which we call $X_{\min}$, is a lower bound for the value of the drag of the combination considered in the problem. We note that this lower bound is not attained in all cases, since regions necessarily appear with negative thickness. In practice, for the reduction of the drag of a wing-fuselage system it is necessary to choose n so that such regions do not exist.

The situation is analogous also for the general case, where the selection of an optimum body is associated with the presence of supplementary conditions of no flow through certain surfaces. Thus the following theorems are demonstrated.

Theorem 1. For the conditions of the problem formulated above a body may in principle be selected such that the total drag experienced by all bodies inside the characteristic surface S differs as little as desired from the lower bound X_{min} for the drag of the combination being studied.

Theorem 2. The distribution of the values of the average potential over the part S_2 of the characteristic surface for the extremal combination agrees with the distribution of potential on S_2 for the equivalent body of revolution.

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